RADIAL DISTRIBUTION OF STRESSES IN A WEDGE AND IN A HALF-PLANE WITH VARIABLE MODULUS OF ELASTICITY

(RADIAL'NOE RASPREDELENIE NAPRIAZHENII V KLINE I POLUPLOSKOSTI S PEREMENNYM MODULEM UPRUGOSTI)

PMM Vol.26, No.1, 1962, pp. 146-151

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(Received July 8, 1961)

We examine the plane elastic problem of an infinite isotropic wedge in which the modulus of elasticity is a continuous function of the coordinates r, θ. The wedge is subjected to a force applied at the apex. We examine in particular the problem of the half-plane. It is established that there exists a definite and rather wide class of functions $E(r, \theta)$ to which there corresponds a so-called radial distribution of stresses ($\sigma_\theta = \tau_{r\theta} = 0$). This distribution differs from that in the wedge and half-plane of constant modulus only by the stress σ_r (in particular cases there is no difference at all).

1. Equation for the modulus of elasticity. We examine an infinite elastic wedge with an apex angle equal to $\beta_1 + \beta_2$ in which Young's modulus E and Poisson's ratio ν are continuous functions of the coordinates r, θ. Here θ is measured from the x-axis, which in general does not coincide with the axis of symmetry. Let the given body be in a state of generalized plane stress or plane strain under the action of a force applied at the apex (Fig. 1). The components of the force, referred to a unit thickness, we denote by P_x, P_y; we use the usual notation for the components of stresses, strains, and displacements.

As is known from the classical linear theory of elasticity, for an isotropic wedge of constant E, ν, there exists a radial distribution of stresses (see, e.g. [1])

$$\sigma_r = (A\cos\theta + B\sin\theta)\frac{1}{r}, \quad \sigma_\theta = \tau_{r\theta} = 0 \qquad (1.1)$$

where A, B are coefficients that are determined by the equilibrium of a portion of the wedge cut out by an arc at an arbitrary radius r

$$\int_{-\beta_1}^{\beta_2} \sigma_r r \cos\theta d\theta = -P_x, \quad \int_{-\beta_1}^{\beta_2} \sigma_r r \sin\theta d\theta = -P_y \tag{1.2}$$

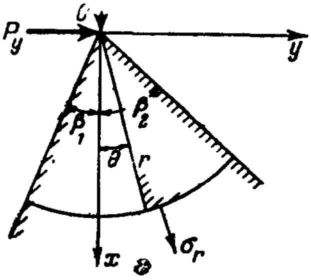

Fig. 1.

Certain results in the plane problem for a body with a variable modulus of elasticity were obtained in the works of Goletskiy [2], Konchkovskiy [3], Teodoresku and Predeleanu [4,5], and others. In the latter two papers a solution was given for a half-plane with a modulus of elasticity in the form of an exponential function of the coordinates and with a periodically repeating load applied on the boundary. The solution of the generally formulated problem for the wedge and half-plane has evidently not yet been found.

Rather than set ourselves the goal of solving the problem in the general case, for arbitrary E, ν, we formulate the problem in the following way: Determine the conditions which Young's modulus and Poisson's ratio must satisfy in order that a radial distribution of stresses ($\sigma_\theta = \tau_{r\theta} = 0$) exists. In this task we shall start from the equations of the linear theory of elasticity.

It is necessary to distinguish two cases – plane strain and generalized plane stress. If in the first case one sets $\sigma_\theta = \tau_{r\theta} = 0$ and introduces new elastic constants

$$E' = \frac{E}{1-\nu^2}, \qquad \mu = \frac{\nu}{1-\nu} \tag{1.3}$$

then the basic system of the equations of equilibrium of the elastic body may be written in the form:

$$\begin{gathered} \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} = 0, \qquad \frac{\partial u}{\partial r} = \frac{\sigma_r}{E'} \\ \frac{\partial v}{\partial \theta} = -\frac{\mu r \sigma_r}{E'} - u, \qquad \frac{\partial u}{\partial \theta} + r\frac{\partial v}{\partial r} - v = 0 \end{gathered} \tag{1.4}$$

From the first three equations we find

$$\sigma_r = \frac{f(\theta)}{r} \tag{1.5}$$

$$u = \int \frac{f(\theta)}{E' r} dr + u', \qquad v = -\int \left[\frac{\mu f(\theta)}{E'} + \int \frac{f(\theta)}{E' r} dr\right] d\theta + v' \tag{1.6}$$

where $f(\theta)$ is an unknown function, and u', v' are "rigid body" displacements. From the fourth equation (1.4) we obtain the condition of compatibility of the system of three equations for the two unknowns u, v, which after elementary transformations takes on the form

$$\frac{\partial^2}{\partial\theta^2}\frac{f}{E'}+\frac{f}{E'}-r\frac{\partial}{\partial r}\frac{f}{E'}-r^2\frac{\partial^2}{\partial r^2}\frac{\mu f}{E'}=0 \tag{1.7}$$

Hence, the stress distribution will be radial in all cases in which E', μ satisfy Equation (1.7). Also entering into this equation is the unknown function f of the single variable θ. This function must satisfy the equilibrium conditions (1.2). In the case of generalized plane stress we obtain the same Expressions (1.6) and Equations (1.7), but it is necessary to replace E'; μ by E, ν

2. Case of constant Poisson's ratio. If the modulus E is a function of r, θ while ν is a constant, then Equation (1.7) becomes

$$\frac{\partial^2}{\partial\theta^2}\frac{f}{E}+\frac{f}{E}-r\frac{\partial}{\partial r}\frac{f}{E}-\mu r^2\frac{\partial^2}{\partial r^2}\frac{f}{E}=0 \tag{2.1}$$

where in the case of plane strain $\mu=\nu/(1-\nu)$ and in the case of generalized plane stress $\mu=\nu$.

We look first for the class of solutions of Equation (2.1) in the form of an arbitrary function of the distance r times a function of the polar angle θ

$$E=E_r(r)\,E_\theta(\theta) \tag{2.2}$$

Substituting into (2.1) and separating the variables, we obtain

$$\left(\frac{f}{E_\theta}\right)''+n^2\frac{f}{E_\theta}=0,\qquad \mu\left(\frac{1}{E_r}\right)''+\frac{1}{r}\left(\frac{1}{E_r}\right)'+\frac{n^2-1}{r^2}\frac{1}{E_r}=0 \tag{2.3}$$

where n is an arbitrary number, real, imaginary, or zero. Integrating, we obtain for $n\neq 0$

$$\frac{f}{E_\theta}=A\cos n\theta+B\sin n\theta,\qquad \frac{1}{E_r}=C_1r^\alpha+C_2r^{-\alpha+1-1/\mu} \tag{2.4}$$

$$E=\frac{E_\theta}{C_1r^\alpha+C_2r^{-\alpha+1-1/\mu}}\qquad\left(\alpha\neq 1,\ \alpha\neq\frac{1}{\mu}\right) \tag{2.5}$$

Here A, B, C_1, C_2 are arbitrary constants. For a given α

$$n=\sqrt{(1-\alpha)(1+\mu\alpha)} \tag{2.6}$$

The quantity E_θ may be an arbitrary function of θ (however it is to be understood that the Expression (2.5) has physical meaning only when it is positive because we always have $E>0$).

If the material has a modulus of elasticity whose analytic form is some particular case of Formula (2.5), then a radial distribution of stresses obtains in the wedge in the form

$$\sigma_r = \frac{E_\theta}{r}(A\cos n\theta + B\sin n\theta), \qquad \sigma_\theta = \tau_{r\theta} = 0 \tag{2.7}$$

The coefficients A, B are found from the conditions of equilibrium (1.2).

The number n will be real for $-1/\mu < a < 1$ and imaginary for $a > 1$ and $a < -1/\mu$. For imaginary n the cosine and sine in (2.7) must be changed to the hyperbolic cosine and sine of argument $in\theta$. In Formula (2.5) the quantity a may also be a complex number, namely

$$\alpha = -\frac{1-\mu}{2\mu} + \delta i, \qquad -\alpha + 1 - \frac{1}{\mu} = -\frac{1-\mu}{2\mu} - \delta i \tag{2.8}$$

where δ is an arbitrary positive number. The corresponding Young's modulus is given by the function

$$E = \frac{E_\theta}{C_1\cos(\delta\ln r) + C_2\sin(\delta\ln r)}\, r^{\frac{1-\mu}{2\mu}} \tag{2.9}$$

The stress σ_r is determined by the Formula (2.7), in which

$$n = \sqrt{\mu\delta^2 + \frac{(1+\mu)^2}{4\mu}} \tag{2.10}$$

For $\nu = 0$, $\mu = 0$ the modulus which insures a radial stress distribution is obtained as the limit of Expression (2.5) (for $C_1 \neq 0$)

$$E = E_\theta r^{-\alpha} \qquad (n = \sqrt{1-\alpha}) \tag{2.11}$$

In addition to (2.5) there also exists the modulus (corresponding to $n = 0$)

$$E = E_\theta\left(C_1 r + C_2 r^{-\frac{1}{\mu}}\right)^{-1} \tag{2.12}$$

for which one obtains a radial distribution of the form

$$\sigma_r = \frac{E_\theta}{r}(A + B\theta), \qquad \sigma_\theta = \tau_{r\theta} = 0 \tag{2.13}$$

For constant E (E_θ = const, $C_2 = a = 0$ in Formula (2.5)) we obtain the well known solution (1.1).

If the modulus E is the sum of expressions of the form of (2.5) for various a, then the stress distribution will no longer be radial (since it is then not possible to determine $f(\theta)$ from (2.1)). However, special cases may be given where the function $1/E$ is in the form of the sum of products of the inverses of (2.5) and where the stress distribution is radial (valid for particular cases of the wedge and the load).

Case 1. The modulus of elasticity is given in the form of the function

$$E = E_\theta \left[C_{10} r + C_{20} r^{-\frac{1}{\mu}} + \sum_{k=1}^{N} \left(C_{1k} r^{\alpha_k} + C_{2k} r^{-\alpha_k + 1 - \frac{1}{\mu}} \right) \cos n_k \theta \right]^{-1} \tag{2.14}$$

Here θ is measured from the x axis, which coincides with the symmetry axis of the wedge ($\beta_1 = \beta_2 = \beta$), C_{1k}, C_{2k} are constants that are not simultaneously zero, E_θ is an even function of θ and moreover such that $E > 0$ in the wedge region

$$n_k = \sqrt{(1 - \alpha_k)(1 + \mu\alpha_k)} \tag{2.15}$$

The stress distribution caused by a force directed along the symmetry axis ($P_y = 0$) is given by the formula

$$\sigma_r = \frac{AE_\theta}{r}, \qquad \sigma_\theta = \tau_{r\theta} = 0 \tag{2.16}$$

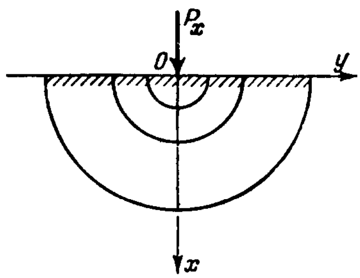

Fig. 2.

The coefficient A is determined from the first of conditions (1.2); the second is satisfied identically. For $\beta = \pi/2$ we obtain the solution for a half-plane under the action of a normal force P_x (Fig. 2). If the quantity E_θ is constant in Formula (2.14), then the solution has the form

$$\sigma_r = -\frac{P_x}{2r}, \qquad \sigma_\theta = \sigma_{r\theta} = 0 \tag{2.17}$$

Level lines of stress σ_r = const (isobars) have the form of circular arcs whose centers are at the point of application of the force (Fig. 2).

Case 2. The modulus of elasticity is given in the form

$$E = E_\theta \left[\left(C_{10} r + C_{20} r^{\frac{1}{\mu}} \right) \theta + \sum_{k=1}^{N} \left(C_{1k} r^{\alpha_k} + C_{2k} r^{-\alpha_k + 1 - \frac{1}{\mu}} \right) \sin n_k \theta \right]^{-1} \tag{2.18}$$

Here θ is measured from the symmetry axis of the wedge, E_θ is an odd function of θ but such that $E > 0$ over the entire wedge region. The stress distribution caused by a force P_y normal to the wedge axis ($P_x = 0$) is given by Formula (2.16).

The first of the equilibrium conditions (1.2) is satisfied identically, while the second serves to determine the constant A.

If there are terms with $\alpha_k > 1$ and $\alpha_k < 1/\mu$ in the sums (2.14) and (2.18), then the corresponding sines and cosines should be changed to hyperbolic functions in the argument $in\theta$.

3. Stress distribution when E obeys a power law. We examine the case of a modulus of elasticity varying according to the law

$$E = E_m x^m = E_m r^m \cos^m \theta \tag{3.1}$$

here m is an arbitrary real number, positive or negative, whole or functional (x is measured from an axis which in general is not perpendicular to the symmetry axis of the wedge). We have a particular case of the Expression (2.5), in which $C_1 = 1$, $C_2 = 0$, $a = -m$, $E_\theta = E_m \cos^m\theta$; the constant

$$n = \sqrt{(1+m)(1-\mu m)} \tag{3.2}$$

will be real for $-1 < m < 1/\mu$ and imaginary for $m < -1$ and $m > 1/\mu$. The stress σ_r is given by the formula

$$\sigma_r = \frac{\cos^m\theta}{r}(A\cos n\theta + B\sin n\theta) \tag{3.3}$$

We obtain for the coefficients A, B the equations

$$\begin{aligned} A\int \cos n\theta \cos^{m+1}\theta\, d\theta + B\int \sin n\theta \cos^{m+1}\theta\, d\theta &= -P_x \\ A\int \cos n\theta \cos^{m}\theta \sin\theta\, d\theta + B\int \sin n\theta \cos^{m}\theta \sin\theta\, d\theta &= -Py \end{aligned} \tag{3.4}$$

(the limits of integration are $-\beta_1$ and β_2).

In the case of imaginary $n = in_1$

$$\sigma_r = \frac{\cos^m\theta}{r}(A\cosh n_1\theta + B\sinh n_1\theta) \tag{3.5}$$

We consider two particular cases of the half-plane.

Case 1. The modulus of elasticity is proportional to the distance from the boundary

$$E = E_1 x, \quad m = 1, \quad \alpha = -1$$

On the basis of Formula (3.3) and Equations (3.4) we find

$$\sigma_r = -\frac{1+\mu}{\sin(n\pi/2)}\frac{\cos\theta}{r}(0.5nP_x\cos n\theta + P_y\sin n\theta) \qquad (n \doteq \sqrt{2(1-\mu)}) \tag{3.6}$$

Since Poisson's ratio varies between the limits 0 to 0.5 for various materials, we have for generalized plane stress that $1 \leqslant n \leqslant \sqrt{2} = 1.4142$.

Likewise for plane strain $0 \leqslant \mu \leqslant 1$ and $0 \leqslant n \leqslant \sqrt{2}$, where $n = 0$ corresponds to an incompressible material ($\nu = 0.5$). In the latter case Formula (3.6) loses validity because the expression for E should be considered as a particular case of Expression (2.12), to which there corresponds the stress (2.13) or

$$\sigma_r = -\frac{2\cos\theta}{\pi r}(P_x + 2P_y\theta) \tag{3.7}$$

Hence it is seen that the stress distribution under the action of a normal load ($P_y = 0$) in the case of plane strain and an incompressible material is obtained in exactly the same form as in the case of a homogeneous isotropic half-plane. We note that this also takes place when the modulus of elasticity depends on the coordinates in the following manner:

$$E = E_1 \cos\theta \left[C_1 r + C_2 r^{\frac{1}{\mu}}\right]^{-1} \tag{3.8}$$

In this case it is easily verified by the use of Formula (2.14) and by determining the constants A and B from the equilibrium conditions (1.2) that for $P_y = 0$

$$B = 0$$

In the case of generalized plane stress and an incompressible material, $n = 1$ and

$$\sigma_r = -0.75\, P_x \frac{\cos^2\theta}{r} \tag{3.9}$$

Level lines of stress (isobars) have an oval shape (Fig. 3).

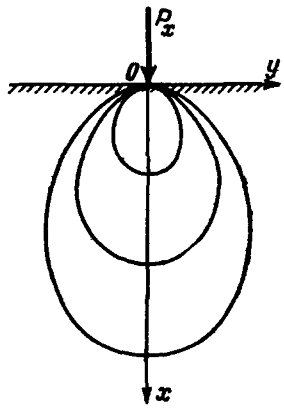

Fig. 3.

Case 2. The modulus of elasticity is inversely proportional to the distance

$$E = \frac{E_{-1}}{x}$$

The expression for E is a particular case of (2.12)

$$C_1 = 1, \quad C_2 = 0, \quad E_\theta = \frac{E_{-1}}{\cos\theta}$$

For a wedge whose x axis coincides with the symmetry axis ($\beta_1 = \beta_2 = \beta$), we obtain

$$\sigma_r = -\frac{1}{2r\cos\theta}\left[\frac{P_x}{\beta} + \frac{P_y}{g(\beta)}\theta\right]$$

$$\left(g(\beta) = \int_0^\beta \theta \tan\theta \, d\theta\right) \tag{3.10}$$

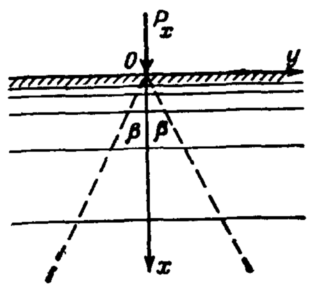

Fig. 4.

As the angle β approaches $\pi/2$, the quantity g grows without bounds and we obtain for the half-plane

$$\sigma_r = -\frac{P_x}{\pi x} \tag{3.11}$$

Hence the theory (at least linear) leads to the conclusion: For a half-plane with a modulus of elasticity which is inversely proportional to the distance from the boundary and with a constant Poisson's ratio, a force directed along the boundary will not give rise to any stresses. Stresses from a normal force vary according to the same law as the modulus E so that the isobars σ_r = const are straight lines parallel to the boundary (Fig. 4).

BIBLIOGRAPHY

1. Timoshenko, S.P., *Teoria Uprugosti (Theory of Elasticity)*. ONTI, 1937.

2. Goletskiy, J., On the foundations of the theory of elasticity of plane incompressible nonhomogeneous bodies. *Arch. mech. stosowanej* Vol. 11, No. 4, 1959.

3. Konchkovskiy, Z., Statics of nonhomogeneous rectangular plates and discs. *Nonhomogeneity in Elasticity and Plasticity*. Pergamon Press, 1959.

4. Teodoresku, P.P. and Predeleanu, M., *Quelques conditions sur le probleme des corps elastiques heterogens*. Pergamon Press, 1959.

5. Teodoresku, P.P. and Predeleanu, M., Uber das ebene Problem Nichthomogener elastischer Korper. *Acta techn. Acad. scient. hung.* Vol. 27, Nos. 3-4, 1959.

Translated by E.E.Z.